Almost Classical Skew Bracoids and the Yang-Baxter Equation

Isabel Martin-Lyons

Keele University, UK

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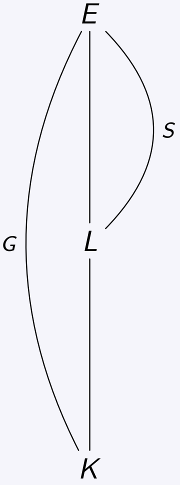
Aim

We will explore some concepts from Hopf-Galois theory as they pertain to skew bracoids and solutions to the Yang-Baxter equation.

Let E/K be a finite Galois extension of fields with L some intermediate field, so that L/K is separable but not necessarily Galois. Write $(G, \circ) = \mathrm{Gal}(E/K)$ and $S = \mathrm{Gal}(E/L)$.



Recall that Hopf-Galois structures on such extensions correspond to skew bracoids; the Galois group occurs as the multiplicative group and the type is encoded in the additive group.

Outline

Outline

Skew braces and bracoids

Definition

A *skew (left) brace* is a triple $(G, \star, \circ)$, where $(G, \star)$ and $(G, \circ)$ are groups and for all $g, h, f \in G$

$$g \circ (h \star f) = (g \circ h) \star g^{-1} \star (g \circ f).$$

Definition

A *skew (left) bracoid* is a 5-tuple $(G, \circ, N, \star, \odot)$, where $(G, \circ)$ and $(N, \star)$ are groups and $\odot$ is a transitive action of G on N for which

$$g \odot (\eta \star \mu) = (g \odot \eta) \star (g \odot e_N)^{-1} \star (g \odot \mu),$$

for all $g \in G$ and $\eta, \mu \in N$.

Housekeeping

- We will frequently write $(G, N, \odot)$, or even (G, N), for $(G, \circ, N, \star, \odot)$.
- We will refer to $(N, \star)$ as the additive group and $(G, \circ)$ as the multiplicative or acting group.
- Any identity will be denoted e, possibly with a subscript.

For example

Examples

- If $(G, \circ)$ is a group then $(G, \circ, \circ)$ and $(G, \circ^{op}, \circ)$ are skew braces, the so-called *trivial* and *almost trivial* skew braces on G.
- Any skew brace $(G, \star, \circ)$ can be thought of as a skew bracoid $(G, \circ, G, \star, \odot)$, where $\odot$ is simply $\circ$. If (G, N) is a skew bracoid with $\mathrm{Stab}_G(e_N)$ trivial we say that (G, N) is *essentially a skew brace*, since we can use the bijection $g \mapsto g \odot e_N$ to transfer the operation in G onto N to give a skew brace on N.
- For any group (G) we have the skew bracoid $(G, \{e\}, \odot)$ where of course the action $\odot$ is trivial.

Something more concrete

Examples

- Let $d, n \in \mathbb{N}$ such that $d|n$. Take $G = \langle r, s \mid r^n = s^2 = e, srs^{-1} = r^{-1} \rangle \cong D_n$ and $N = \langle \eta \rangle \cong C_d$. Then we get a skew bracoid $(G, N, \odot)$ for $\odot$ given by

$$r^i s^j \odot \eta^k = \eta^{i+(-1)^j k}.$$

Reduction

Definition

We say that a skew bracoid $(G, N, \odot)$ is *reduced* if and only if the action $\odot$ of G on N is faithful.

From a Hopf-Galois perspective, this means that the "top" field was really the Galois closure of the one we are interested in.

Proposition

Let $(G, N, \odot)$ be a skew bracoid, and K be the kernel of the map $\mathcal{L}_\odot : G \to \mathrm{Hol}(N)$. We can form a skew bracoid $(G/K, N, \odot_K)$ which is reduced and $\mathcal{L}_\odot$ and $\mathcal{L}_{\odot_K}$ have the same image in $\mathrm{Hol}(N)$.

Reduction examples

Examples

- Any skew bracoid that is essentially a skew brace is reduced since the action of the multiplicative group on the additive is regular and therefore faithful.
- Skew bracoids of the form $(G, \{e\}, \odot)$ are not reduced (unless G is itself trivial), since all of G is in $K = \ker(\mathcal{L}_\odot)$.
- In $(G, N) \cong (D_n, C_d)$, we see that (G, N) is reduced if and only if $d = n$, since $\langle r^d \rangle \subseteq K$ as $r^d \odot \eta^k = \eta^{d+k} = \eta^k$.

Outline

Containing a brace

Definition

We say that a skew bracoid (G, N) *contains a brace* if the subgroup $S = \mathrm{Stab}_G(e_N)$ has a complement H in G, so that $G = HS$.

This is equivalent to saying that G contains a subgroup H for which (H, N) is essentially a skew brace.

Definition

A skew bracoid (G, N) is *almost a skew brace* if the subgroup $S = \mathrm{Stab}_G(e_N)$ has a normal complement in G, so that $G = HS \cong H \rtimes S$.

Almost classical skew bracoids

Definition

A skew bracoid (G, N) is *almost classical* if the subgroup $S = \mathrm{Stab}_G(e_N)$ has a normal complement H in G, and when thought of as a skew brace, (H, N) is trivial.

This is saying that when the operation in H is transferred to N, it coincides with the original operation in N. Explicitly this means, for all $h_1, h_2 \in H$

$$(h_1 \odot e_N)(h_2 \odot e_N) = h_1 h_2 \odot e_N,$$

and consequently

$$(h_1 \odot e_N)^{-1} = h_1^{-1} \odot e_N.$$

Reduction

Proposition

Let (G, N) be a skew bracoid that contains a brace (resp. is almost a brace, is almost classical), then its reduced form $(G/K, N)$ also contains a brace (resp. is almost a brace, is almost classical).

Intuitively, when we quotient by K we only kill off part of the stabiliser, so our favourite H is preserved and we have $(H/K, N)$ as essentially a skew brace in $(G/K, N)$.

Unfortunately, the converse does not hold in general.

Almost and almost classical

Remark

Let (G, N) be a reduced skew bracoid and suppose that there is some $H \subseteq G$ for which (H, N) is essentially a trivial skew brace, then (G, N) is almost a brace.

- Immediately, we have that H is a complement to $S = \mathrm{Stab}_G(e_N)$ since H is regular on N.
- Normality is easiest to see via the holomorph, note that since (G, N) is reduced the map $\mathcal{L}_\odot$ is injective. For $h \in H$ we have that $\mathcal{L}_\odot(h)$ coincides with $\mathcal{L}_\star(h \odot e_N)$, so $\mathcal{L}_\odot(H) = \mathcal{L}_\star(N)$ as permutation groups. Then $\mathcal{L}_\star(N)$ is a normal subgroup of $\mathrm{Hol}(N)$ and therefore of G embedded into $\mathrm{Hol}(N)$. Finally, as $G \cong \mathcal{L}_\odot(G)$ we have that $H \cong \mathcal{L}_\odot(H) = \mathcal{L}_\star(N)$ is normal in G.

For example

Examples

- When thought of as skew bracoids, every skew brace is almost a skew brace, and every trivial skew brace is almost classical.
- Skew bracoids of the form $(G, \{e\}, \odot)$ are almost classical since they contain the brace $(\{e\}, \{e\})$, $\{e\}$ is normal in G and $(\{e\}, \{e\})$ is certainly trivial.
- Take $(G, N) \cong (D_n, C_n)$, then $S = \langle s \rangle$ since $s^j \odot \eta^0 = \eta^{(-1)^j \cdot 0}$ and $r^i \odot e_N = \eta^i$ so it is clear that $(\langle r \rangle, N)$ is trivial. We know $G \cong \langle r \rangle \rtimes \langle s \rangle$, hence (G, N) is almost classical.
- Recall Byott's example $(G, N) \cong (GL_3(\mathbb{F}_2), C_2 \times C_2 \times C_2)$ which we know to contain a brace, however it cannot be almost a skew brace since G is simple.

Induced skew bracoids

Theorem

Let $(G, N, \odot_N)$ be a skew bracoid that is almost a skew brace due to H and write S for $\mathrm{Stab}_G(e_N)$. Suppose we also have a skew bracoid $(S, M, \odot_M)$. We may take the (external) direct product of N and M and, define

$$hs \odot (\eta, \mu) := (hs \odot_N \eta, s \odot_M \mu)$$

for $h \in H$, $s \in S$ and $(\eta, \mu) \in N \times M$. This $\odot$ is then an action, under which $(G, N \times M, \odot)$ is a skew bracoid.

Observe that to act on the M component, we are using the natural projection of G onto S.

Inducing a skew bracoid that contains a brace

Proposition

Let (G, N) and (S, M) be skew bracoids with $G = HS \cong H \rtimes S$. If (S, M) contains a brace then so does the induced skew bracoid $(G, N \times M)$.

Proof.

Say (S, M) contains the brace (T, M). Then HT is a subgroup of G, and

$$ht \odot (e_N, e_M) = (h \odot_N e_N, t \odot_M e_M). \tag{1}$$

Note H is transitive on N and T is transitive on M so HT is transitive on $N \times M$. The right hand side of (1) is (e_N, e_M) if and only if $h = e_G$ and $t = e_G$, since both (H, N) and (T, M) are essentially skew braces. Hence $(HT, N \times M)$ is essentially a skew brace and $(G, N \times M)$ contains a brace. □

Outline

The γ-function

Definition/Proposition

Given a skew bracoid $(G, \circ, N, \star, \odot)$, we define the map $\gamma : (G, \circ) \to \mathrm{Perm}(N, \star)$, sending g to γ_g, by

$$\gamma_g(\eta) = (g \odot e_N)^{-1} \star (g \odot \eta),$$

for $g \in G$ and $\eta \in N$.

Then γ is in fact a homomorphism, with image in $\mathrm{Aut}(N, \star)$. We call this map the γ-*function* of the skew bracoid.

Recall that this is the backbone of the route from a skew bracoid to a solution to the Yang-Baxter equation.

The γ-function of an almost classical skew bracoid

Let (G, N) be an almost classical skew bracoid with H such that $G \cong H \rtimes S$ and (H, N) is essentially a trivial skew brace.

Let $h \in H$ and $s \in S$, and $\eta = h_\eta \odot e_N \in N$ then,

$$\begin{aligned} \gamma_{hs}(\eta) &= (hs \odot e_N)^{-1} \star (hs \odot \eta) \\ &= (h \odot e_N)^{-1} \star (hsh_\eta \odot e_N) \\ &= (h^{-1} \odot e_N) \star (hsh_\eta s^{-1} \odot e_N) \\ &= sh_\eta s^{-1} \odot e_N. \end{aligned}$$

So we conjugate by the S part and the H part acts trivially.

Example

In the $G = \langle r, s \rangle \cong D_n$ acting on $N = \langle \eta \rangle \cong C_n$ example we have $\gamma_{r^i s^j}(\eta^k) = s^j r^k s^{-j} \odot e_N = \eta^{(-1)^j k}$.

The Yang-Baxter Equation

Definition

A *solution to the set-theoretic Yang-Baxter equation* (hereafter simply a *solution*) is a non-empty set G, together with a map $r : G \times G \to G \times G$ satisfying

$$(r \times 1)(1 \times r)(r \times 1) = (1 \times r)(r \times 1)(1 \times r)$$

as functions on $G \times G \times G$.

Given a solution r on G, for all $x, y \in G$ we write

$$r(x, y) = (\lambda_x(y), \rho_y(x));$$

so that we have family of maps $\lambda_x : G \to G$ and a family of maps $\rho_y : G \to G$.

Properties of the Solution

Suppose G with r is a solution and write $r(x, y) = (\lambda_x(y), \rho_y(x))$.

We say this solution is:

- *bijective* if r is bijective;
- *left non-degenerate* if λ_x is bijective for all $x \in G$;
- *right non-degenerate* if ρ_y is bijective for all $y \in G$;
- *non-degenerate* if r is both left and right non-degenerate.

Example

- The trivial solution $r(x, y) = (x, y)$ is bijective and degenerate.
- The twist solution $r(x, y) = (y, x)$ is bijective and non-degenerate.

Solutions from skew bracoids

Let (G, N) be a skew bracoid that contains a brace (H, N). We have that the map $a : h \mapsto h \odot e_N$ is a bijection between H and N, we write b for its inverse. Recall that with this we can define

$$\lambda_x(y) = b(\gamma_x(y \odot e_N))$$

and then

$$\rho_y(x) = \lambda_x(y)^{-1}xy,$$

for all $x, y \in G$.

This λ and ρ form a Lu-Yan-Zhu pair so that G with $r(x, y) = (\lambda_x(y), \rho_y(x))$ forms a (right non-degenerate but possibly left degenerate) solution.

Two sub-solutions

Given this setup, we have that

- H with r is a bijective non-degenerate solution - the one coming from the (essentially a) skew brace (H, N);
- and restricting to S we have

$$\lambda_{s_1}(s_2) = b(\gamma_{s_1}(s_2 \odot e_N)) = e_G, \qquad \rho_{s_2}(s_1) = s_1 s_2.$$

for $s_1, s_2 \in S$, so we get an entirely left degenerate sub-solution.

Reduction in solutions

Consider the subgroup $K = \ker(\mathcal{L}_{\odot})$ of G. Simply restricting to K, we get the relations are the same as in S, meaning K with r is also a solution. But moreover, for $k \in K$ and all $hs \in G$, we have

$$\lambda_k(hs) = h, \qquad \rho_{hs}(k) = h^{-1}khs,$$
$$\lambda_{hs}(k) = e_G, \qquad \rho_k(hs) = hsk.$$

Almost classical solutions

Suppose (G, N) is almost classical due to a subgroup H of G, i.e. (H, N) is essentially a trivial skew brace.

Taking $h_1, h_2 \in H$ and $s_1, s_2 \in S$, the solution arising from (G, N) is the given by

$$\begin{aligned}\lambda_{h_1 s_1}(h_2 s_2) &= b(\gamma_{h_1 s_1}(h_2 \odot e_N)) \\ &= b(s_1 h_2 s_1^{-1} \odot e_N) \\ &= s_1 h_2 s_1^{-1},\end{aligned}$$

and

$$\begin{aligned}\rho_{h_2 s_2}(h_1 s_1) &= s_1 h_2^{-1} s_1^{-1} h_1 s_1 h_2 s_2 \\ &= s_1 h_2^{-1} s_1^{-1} h_1 s_1 h_2 s_1^{-1} s_1^{-1} s_1 s_2 \\ &= \lambda_{h_1 s_1}(h_2 s_2)^{-1} h_1 \lambda_{h_1 s_1}(h_2 s_2) s_1 s_2.\end{aligned}$$

Which contains...

The solution arising from (G, N) contains the solution associated to the trivial skew brace (H, N). Recall that γ_h is trivial for all $h \in H$ so restricting to H we recover the solution given by

$$\lambda_{h_1}(h_2) = h_2, \qquad \rho_{h_2}(h_1) = h_2^{-1} h_1 h_2,$$

which is a solution coming from the group H.

Running Example

Example

In our $(G, N) \cong (D_n, C_n)$ example

$$\begin{aligned}\lambda_{r^i s^j}(r^k s^\ell) &= b(\gamma_{r^i s^j}(\eta^k)) \\ &= b(\eta^{(-1)^j k}) \\ &= r^{(-1)^j k}.\end{aligned}$$

From this we get

$$\begin{aligned}\rho_{r^k s^\ell}(r^i s^j) &= \lambda_{r^i s^j}(r^k s^\ell)^{-1} r^i s^j r^k s^\ell \\ &= r^{-(-1)^j k} r^{i+(-1)^j k} s^{j+\ell} \\ &= r^i s^{j+\ell}.\end{aligned}$$

Hence $r(r^i s^j, r^k s^\ell) = (r^{(-1)^j k}, r^i s^{j+\ell})$ is a solution.

A note on induced skew bracoids

As we said, if we have a skew bracoid (G, N) that is almost a skew brace (H, N) and a skew bracoid (S, M) that contains a brace (T, M) then we can induce a skew bracoid $(G, N \times M)$ that contains the brace $(HT, N \times M)$.

In the context of solutions this is adding to the non-degenerate piece, and if you like shrinking the (left) degenerate piece. If (S, M) is essentially a skew brace, we would have induced a skew bracoid that was essentially a skew brace and hence get a bijective non-degenerate solution.

Open Questions

- In our induced construction, in some sense, we are taking a product of (H, N) and (S, M), using the fact that H and S are special subgroups of G in (G, N) to ensure that all the necessarily relations hold. Can we extract what these conditions are in order to generalise it? Is this "just" a special case of a matched product? Can we stop at semi-direct product on the way?
- In the case where the skew bracoid is almost a skew brace, the solution coming from the skew brace it contains should relate somehow nicely with the degenerate piece on S. But nice in what way?
- How is the solution on a skew bracoid related to the solution on its reduced form?

Thank you for your attention!